Math 10B with Professor Stankova
Worksheet, Discussion #4; Tuesday, 2/5/2019
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Binomial Identities

Concepts

1. We can write $C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$. One basic identity we have is the **binomial theorem** which says
$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$
There are other equalities that can be proven either algebraically or combinatorially; by counting the same team making strategy in two different ways.

Examples

2. Show that $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.

Solution: First, we can do a combinatorial reasoning. Suppose that I have n people and I want to choose an r person team and inside that team, a k person special committee. One way to count the number of ways to do this is first I choose the r person team, and I can do this $\binom{n}{r}$ ways. Then, from within this team, I choose the committee and this can occur in $\binom{r}{k}$ ways. Thus, this gives a total of $\binom{n}{r}\binom{r}{k}$ ways.

Another way to do this is first choose the committee and this can occur in $\binom{n}{k}$ ways. After I choose the committee, I can choose the remaining people on the team and there are $r-k$ people left to choose. But now there are only $n-k$ people left to choose from since we chose k of them for the committee. Thus, we can complete the team in $\binom{n-k}{r-k}$ ways. This gives a total of $\binom{n}{k}\binom{n-k}{r-k}$ different ways to do this. Since these are two ways of counting the same number of ways, they are equal and
$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$
Algebraically, we have
$$\binom{n}{r}\binom{r}{k} = \frac{n!}{r!(n-r)!}\frac{r!}{k!(r-k)!} = \frac{n!}{(n-r)!k!(r-k)!}.$$
And the other term is
$$\binom{n}{k}\binom{n-k}{r-k} = \frac{n!}{k!(n-k)!}\frac{(n-k)!}{(r-k)!(n-k-(r-k))!} = \frac{n!}{k!(r-k)!(n-r)!} = \binom{n}{r}\binom{r}{k}.$$

3. Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

> **Solution:** The binomial theorem tells us that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Let $x = 1$, then we get the desired equality.

Problems

4. **TRUE** False $\displaystyle\sum_{k=1}^{100} k\binom{100}{k} = 100 \cdot 2^{99}$.

> **Solution:** This is an example of the binomial equality $\displaystyle\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

5. Prove that $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

> **Solution:** Plug in $x = 2$ to the binomial theorem $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

6. What is the coefficient of x^2y^3 in $(2x - 3y)^5$?

> **Solution:** The coefficient is $\binom{5}{2}(2x)^2(-3y)^3 = 10 \cdot 2^2 \cdot (-3)^3 = -1080$.

7. Prove that $k\binom{n}{k} = n\binom{n-1}{k-1}$ in two different ways.

> **Solution:** Algebraically, we play with factorials to get
>
> $$k\binom{n}{k} = \frac{n!k}{k!(n-k)!} = \frac{n!k}{k(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!}$$
>
> and
>
> $$n\binom{n-1}{k-1} = \frac{n(n-1)!}{(k-1)!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = k\binom{n}{k}.$$
>
> Then combinatorially, we can think of this as creating a team of k people out of n total with a designated captain. We can first pick our team in $\binom{n}{k}$ ways then choose a captain out of the k people which gives $k\binom{n}{k}$ ways. The other way is to first pick the captain then pick the rest of the team which is $k-1$ people out of the $n-1$ remaining people which gives $n\binom{n-1}{k-1}$. Since both count the same thing, they must be equal.

8. What is the coefficient of x^4y^9 in $(2x^2+5y^3)^5$?

Solution: A general term is $\binom{5}{k}(2x^2)^k(5y^3)^{5-k} = \binom{5}{k}2^k5^{5-k}x^{2k}y^{3(5-k)}$. Matching powers for x, y we get $2k = 4$ and $3(5-k) = 9$ which both lead to $k = 2$. Thus, the coefficient is

$$\binom{5}{2}2^25^{5-2} = 10 \cdot 4 \cdot 5^3 = 5000.$$

9. (Challenge) What is the coefficient of $x^2y^2z^2$ in $(x+y+z)^6$?

Solution: We can write this as $(x+(y+z))^6$ and so for the term with x^2, we have $\binom{6}{2}x^2(y+z)^4$. The term with y^2z^2 in $(y+z)^4$ is $\binom{4}{2}y^2z^2$ so the term with $x^2y^2z^2$ is $\binom{6}{2}\binom{4}{2} = 90$.

Permutations and Combinations

Examples

10. How many ways can 6 people play in 3 tennis matches if the matches occur at different times? If they occur at the same time (and are indistinguishable)?

Solution: If it is three different times, there are $\binom{6}{2}$ ways for the first match, $\binom{4}{2}$ for the second and $\binom{2}{2}$ choices for the last so a total of

$$\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{6!}{2!2!2!}.$$

If they occur at the same time, then the order of the three matches does not matter so we need to divide by 3! so our final answer is

$$\frac{6!}{2!2!2!3!}.$$

Problems

11. How many ways are there to rearrange the letters of ZYZZYX?

Solution: There are 3 repeated Z's and 2 repeated Y's so $\frac{6!}{3!2!} = 60$.

12. How many ways can we distribute 12 different cookies to 3 people if each person gets 3 (there are 3 left over)?

Solution: We give the first person 3 cookies and there are $\binom{12}{3}$ ways to do this. Then we give the second person 3 and there are $\binom{9}{3}$ ways to do this. Finally, we give the last person 3 and there are $\binom{6}{3}$ ways to do this. So the answer is

$$\binom{12}{3}\binom{9}{3}\binom{6}{3}.$$

13. How many ways can we separate 12 different cookies into 4 piles of 3 if the piles are indistinguishable?

Solution: We first assume that they are distinguishable and using the logic from before, there are

$$\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = \frac{12!}{3!3!3!3!}$$

ways to do this. Then, we need to divide by 4! because the 4 piles are indistinguishable so any ordering we do will be the same. So the final answer is

$$\frac{12!}{3!3!3!3!4!}.$$